Optical solitons in nematic liquid crystals: model with saturation effects

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Abstract

We study a 2-D system that couples a Schrödinger evolution equation to a nonlinear elliptic equation and models the propagation of a laser beam in a nematic liquid crystal. The nonlinear elliptic equation describes the response of the director angle to the laser beam electric field. We obtain results on well-posedness and solitary wave solutions of this system, generalizing results for a well-studied simpler system with a linear elliptic equation for the director field. The analysis of the nonlinear elliptic problem shows the existence of an isolated global branch of solutions with director angles that remain bounded for arbitrary electric field. The results on the director equation are also used to show local and global existence, as well as decay for initial conditions with sufficiently small L^2-norm. For sufficiently large L^2-norm we show the existence of energy minimizing optical solitons with radial, positive and monotone profiles.

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1 Introduction

We present results on the well-posedness and soliton solutions of the coupled nonlinear Schrödinger (NLS) equation

$$\text{(1.1a)} \qquad \mathrm{i}\partial_z u + \frac{1}{2}\nabla^2 u + u\sin(2\theta) = 0,$$

$$\text{(1.1b)} \qquad \nu\nabla^2\theta - q\sin(2\theta) = -2\,|u|^2\cos(2\theta),$$

see [15], where u, and θ depend on the "optical axis" coordinate $z \in \mathbb{R}$, and the "transverse coordinates" $(x, y) \in \mathbb{R}^2$. $\nabla^2 = \partial_x^2 + \partial_y^2$ is the Laplacian in the transverse directions, and ν, q are positive constants.

System (1.1) models the interaction between the complex amplitude u of the electric field of a polarized laser beam propagating through a nematic liquid crystal sample, and the director field angle θ describing the macroscopic orientation of the molecules of the liquid crystal. The variable z plays the role of time, i.e. we are interested in the solutions of (1.1) given $u(x, y, z)$, $\theta(x, y, z)$ at $z = 0$. The experimental set-up described by (1.1) was studied extensively by Assanto and collaborators [16, 4, 15], and is considered one of the first and still few physical systems shown experimentally to support 2-dimensional stable optical solitons [18].

Our first goal is to extend results on the simpler model

$$\text{(1.2a)} \qquad \mathrm{i}\partial_z u + \frac{1}{2}\nabla^2 u + 2\theta u = 0,$$

$$\text{(1.2b)} \qquad \nu\nabla^2\theta - 2q\theta = -2\,|u|^2\,,$$

obtained from (1.1) using the small angle approximation $\sin\theta \approx \theta$, $\cos\theta \approx 1$. The director angle equation (1.2b) has a unique solution $\theta = G * |u|^2$, where $G(x) = 2\nu^{-1}\,N_0(\sqrt{2q/\nu}\,|x|)$ and N_0 is the modified Bessel function, so that (1.2) can be also written as a NLS equation with a Hartree-type nonlinearity

$$\text{(1.3)} \qquad \mathrm{i}\partial_z u + \frac{1}{2}\nabla^2 u + 2\left(G * |u|^2\right)u = 0.$$

Equations (1.2) capture the physical effect that a localized electric field u can produce a deformation of the director angle θ at longer distances. As was recognized by several authors [19, 9], this nonlocal effect regularizes the dynamics of the electric field and avoids the finite-time blow-up seen in the cubic power NLS in two dimensions, see [20]. Specifically, the initial value problem for (1.3) is well-posed [5, 3], and the equation has an energy minimizing soliton solution above a power threshold [14]. Other analytical predictions based on this model have been compared to experimental data and can explain stabilization effects for more complicated structures such as vortices and multisolitons, see e.g. [17, 13]. Variants of (1.3) have been also used to describe optical solitons and related effects in other nonlocal media, often referred to as thermal media [2, 8, 21, 11].

The new feature of (1.1) is the nonlinear equation for the director field, and our first result is that given $u \in L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, equation (1.1b) has a unique solution $\theta(|u|^2)$ that belongs to $H^2(\mathbb{R}^2)$ and satisfies $\theta(x) \in [0, \pi/4)$, for all $x \in \mathbb{R}^2$, see Proposition 3.1. This is a "saturation" effect for the angle. The proof of Proposition 3.1 uses fixed point and continuation arguments, see Lemma 3.4, and also relies on identifying in (1.1b) an analogue of the smoothing operator $(-\nabla^2 + 1)^{-1}$ of (1.2b). The range of θ is deduced from (1.1b) by considering the signs of the trigonometric nonlinearities, see Lemmas 3.1, 3.2.

The results on the director field equation are then used to show local and global existence results for initial conditions $(u_0, \theta(|u_0|^2))$ with $u_0 \in H^1(\mathbb{R}^2)$, see Theorems 4.1, 4.2 respectively for precise statements. The global existence uses the conservation of the L^2-norm of u, and of the Hamiltonian H of the coupled system, where

(1.4)
$$H(u,\theta) = \frac{1}{4}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \nu\, |\nabla \theta|^2 - 2\, |u|^2 \sin(2\theta) + q\,(1 - \cos(2\theta)) \right) dx.$$

The formal functional derivatives of H are

$$\begin{aligned} \partial_u H &= -\frac{1}{2}\nabla^2 u - u \sin(2\theta), \\ \partial_\theta H &= -\frac{\nu}{2}\nabla^2 \theta - |u|^2 \cos(2\theta) + \frac{q}{2}\sin(2\theta), \end{aligned}$$

thus

$$\begin{aligned} \frac{dH}{dz}(u,\theta) &= \langle \partial_u H, \partial_z u\rangle + \langle \partial_\theta H, \partial_z \theta\rangle \\ &= \langle \mathrm{i}\partial_z u, \partial_z u\rangle = 0, \end{aligned}$$

with $\langle \cdot, \cdot \rangle$ the L^2 inner product. The above also imply a formal Hamiltonian structure for (1.1). By (1.4) we have

$$\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \leq 4H(u,\theta) + 3\,\|u\|_{L^2}^2\,, \tag{1.5}$$

and therefore $\|u\|_{H^1}$ should remain bounded for all times.

We also show results on the existence of soliton solutions. The solutions we obtain are minimizers of the Hamiltonian over configurations (u, θ) with

$u \in H^1(\mathbb{R}^2)$ and fixed L^2-norm. The existence statement, Proposition 5.3, assumes that the L^2-norm of u is above certain threshold. We have also used Strichartz estimates to show that if the initial condition $u_0 \in H^1(\mathbb{R}^2)$ has a sufficiently small L^2-norm, then the L^4-norm of the solution $u(t)$ in time must eventually decay to zero, see Proposition 4.3. This implies the nonexistence of solitary wave solutions with small L^2-norm. Our results on energy minimizing soliton solutions are similar to those on (1.3), see [14]. Here, in addition to dealing with the nonlinear elliptic equation and generalizing the Hamiltonian structure, we also simplify the minimization argument by using symmetrization before looking for the minimum. We then use the direct method to show convergence of a subsequence of a minimizing sequence of radial configurations, see Proposition 5.3.

The result that the angle θ is bounded by $\pi/4$ is interesting from the physical point of view. It suggests a saturation effect for optical solitons in liquid crystals that seems not to have received much attention. Saturation (and possibly smoothing) occur in (1.1) even for $\nu = 0$. We include an Appendix where we outline a formal derivation of (1.1) and explain the geometry of the experiment modeled by (1.1), using [12, 7, 6], and [16]. We see that the director field angle with the $z-$axis is $\theta_0 + \theta(x, y, z)$, where θ_0 is a constant "pre-tilt" angle induced by an external "bias" electric field E (θ_0 and E are absorbed by the constants of (1.1b)). The assumption $q > 0$ in (1.1b) is equivalent to $\theta_0 > \pi/4$. Then, our bound is consistent with $\theta_0 + \theta < \pi/2$, i.e. the statement that the director angle always points forward, in the direction of the propagation of the laser beam. The Appendix shows that model (1.1) is also derived under a small angle assumption. The saturation effect thus requires further study, using some of the more general models in the Appendix. On the other hand, (1.1) seems to be the simplest model describing saturation effects in liquid crystal optics.

The paper is organized as follows. In section 2 we present some preliminary results. In section 3 we show existence, unique continuation, and regularity for the director equation (1.1b). The results are summarized in Proposition 3.1. In section 4 we show local and global well-posedness for the initial value problem (1.1). We also show decay for small initial conditions, see Proposition 4.3. In section 4 we show the existence of constrained mininimizers for the Hamiltonian, implying the existence of radially symmetric optical solitons, see Proposition 5.3. In appendix A we give a formal derivation of (1.1).

2 Notation and preliminary results

Let $N : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory conditions

1. $N(.,\theta)$ is measurable for each $\theta \in \mathbb{R}$,
2. $N(x,.)$ is continuous in $\mathbb{R}$ for almost all $x \in \mathbb{R}^2$;

and the inequality $|N(x,\theta)| \leq C\,(|\theta| + \varphi(x))$, with $\varphi \in L^2(\mathbb{R}^2)$. Then the Nemytskii operator defined as $\theta \mapsto N(x,\theta)$ is a continuous bounded map in $L^2(\mathbb{R}^2)$.

Given $v \in L^2(\mathbb{R}^2)$, we define $v^+ = \max(v,0)$ and $v^- = \max(-v,0)$. Then $\|v^\pm\|^2_{L^2} = \pm\langle v, v^\pm\rangle$, $\langle v^+, v^-\rangle = 0$ and $\|v\|^2_{L^2} = \|v^+\|^2_{L^2} + \|v^-\|^2_{L^2}$. If $v \in H^1(\mathbb{R}^2)$, then $v^\pm \in H^1(\mathbb{R}^2)$. Moreover $\|\nabla v^\pm\|^2_{L^2} = \pm\langle \nabla v, \nabla v^\pm\rangle$, $\langle \nabla v^+, \nabla v^-\rangle = 0$ and $\|\nabla v\|^2_{L^2} = \|\nabla v^+\|^2_{L^2} + \|\nabla v^-\|^2_{L^2}$.

Let $\{W(z) : z \in \mathbb{R}\}$ be the unitary group in $L^2(\mathbb{R}^2)$ generated by $\frac{\mathrm{i}}{2}\nabla^2$. $W(z)$ is an isometric isomorphism in $H^s(\mathbb{R}^2)$, for any $s \in \mathbb{R}$. The integral solution of the inhomogeneous problem

$$\begin{cases} \mathrm{i}\partial_z u + \frac{1}{2}\nabla^2 u + f = 0, \\ u(0) = u_0, \end{cases}$$

is written as $u = h + g$, where

$$h(z) = W(z)u_0, \quad g(z) = \mathrm{i}\int_0^z W(z-z')f(z')dz'. \tag{2.1}$$

Let $1 < r \leq 2 \leq p < \infty$, $q = 2p/(p-2)$ and $\gamma = 2r/(3r-2)$. From the Strichartz estimates ([3]) we can see that there exist $C_p, C_{p,r} > 0$ such that

$$\|h\|_{L^q(I,L^p)} \leq C_p \|u_0\|_{L^2}, \tag{2.2}$$

$$\|g\|_{L^q(I,L^p)} \leq C_{p,r} \|f\|_{L^\gamma(I,L^r)}, \tag{2.3}$$

for any interval $I \subset \mathbb{R}$.

We also recall the well-known Gagliardo–Nirenberg inequalities

$$\|v\|_{L^q} \leq C_{p,q,r} \|\nabla v\|^\alpha_{L^p} \|v\|^{1-\alpha}_{L^r}, \tag{2.4}$$

where $\frac{1}{q} - \frac{1}{r} = \left(\frac{1}{p} - \frac{1}{2} - \frac{1}{r}\right)\alpha$ and $0 \leq \alpha \leq 1$, $1 \leq p,q,r \leq \infty$. Then, taking $q = \infty$, $p = 4$, $r = 2$ and $\alpha = 2/3$, we have that

$$\|v\|_{L^\infty} \leq C \|v\|^{1/3}_{L^2} \|\nabla v\|^{2/3}_{L^4}. \tag{2.5}$$

Another case is $q = 4$, $p = 2$, $r = 2$ and $\alpha = 1/2$

$$\|v\|_{L^4} \le C\, \|v\|_{L^2}^{1/2}\, \|\nabla v\|_{L^2}^{1/2}. \tag{2.6}$$

Given $\zeta > 0$, we will consider the Banach space Y_ζ defined by

$$Y_\zeta = \left\{ u \in C([0,\zeta], H^1(\mathbb{R}^2)) : \nabla u \in L^4([0,\zeta], L^4(\mathbb{R}^2)) \right\}, \tag{2.7}$$

with the norm

$$\|u\|_{Y_\zeta} = \|u\|_{C([0,\zeta],H^1(\mathbb{R}^2))} + \|\nabla u\|_{L^4([0,\zeta],L^4(\mathbb{R}^2))}. \tag{2.8}$$

Lemma 2.1. *Let $V \in L^\infty(\mathbb{R}^2)$, with $V \ge 0$ and $\liminf_{|x|\to\infty} V(x) \ge a > 0$. If $Q : H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \to \mathbb{C}$ is given by*

$$Q(\psi, \varphi) = \int_{\mathbb{R}^2} (\nabla\psi \nabla\varphi^* + V\psi\, \varphi^*)\, dx,$$

then there exist $C_1, C_2 > 0$ such that $C_1 \|\psi\|_{H^1}^2 \le Q(\psi,\psi) \le C_2 \|\psi\|_{H^1}^2$.

Proof. It is clear that $Q(\psi,\psi) \le C_2 \|\psi\|_{H^1}^2$, where $C_2 = \max\{1, \|V\|_{L^\infty}\}$.

Suppose, contrary to our claim, that there exists $\{\psi_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^2)$ such that $\|\psi_n\|_{H^1} = 1$ and $\lim_{n\to\infty} Q(\psi_n,\psi_n) = 0$. Let $R > 0$ such that $V(x) \ge a/2$ for $|x| > R$, given $\varepsilon > 0$ there exists n_0 such that

$$\int_{|x|>R} |\psi_n|^2\, dx \le \frac{2}{a} \int_{|x|>R} V\, |\psi_n|^2\, dx \le \frac{2}{a} Q(\psi_n,\psi_n) < \varepsilon.$$

From Theorem 2.32 in [1], there exists a subsequence of $\{\psi_n\}_{n\in\mathbb{N}}$ such that $\psi_n \to \psi$ in $L^2(\mathbb{R}^2)$ and, since $\|\nabla\psi_n\|_{L^2}^2 \le Q(\psi_n,\psi_n) \to 0$, $\psi \in H^1(\mathbb{R}^2)$ with $\nabla\psi = 0$. Since ψ is a constant function, we have $\psi = 0$. But,

$$\|\psi\|_{L^2}^2 = \lim_{n\to\infty} \|\psi_n\|_{L^2}^2 = \lim_{n\to\infty} \|\psi_n\|_{H^1}^2 - \|\nabla\psi_n\|_{L^2}^2 = 1,$$

a contradiction. ∎

Lemma 2.2. *Let V be as in the previous lemma. Then for any $f \in L^2(\mathbb{R}^2)$, there exists a unique solution $\psi \in H^2(\mathbb{R}^2)$ of $-\nabla^2\psi + V\psi = f$. Moreover, there exists a constant $K = K(V) > 0$ such that $\|\psi\|_{H^2} \le K \|f\|_{L^2}$.*

Proof. From Lemma 2.1, Q is a symmetric bilinear form which defines an inner product equivalent to the usual one in H^1. Let μ be the continuous linear functional in $H^1(\mathbb{R}^2)$ given by $\mu(\varphi) = \langle f, \varphi\rangle_{L^2}$. By the Riesz representation theorem, there exists a unique $\psi \in H^1(\mathbb{R}^2)$ such that $Q(\psi,\varphi) = \mu(\varphi)$, which implies that ψ is a weak solution of $-\nabla^2\psi + V\psi = f$, and $\|\psi\|_{H^1} \le C\|f\|_{L^2}$. Moreover, $\nabla^2\psi \in L^2(\mathbb{R}^2)$ and

$$\left\|\nabla^2\psi\right\|_{L^2} \le \|V\psi\|_{L^2} + \|f\|_{L^2} \le C\,(1 + \|V\|_{L^\infty})\, \|f\|_{L^2},$$

which implies $\|\psi\|_{H^2} \le K\|f\|_{L^2}$. ∎

3 Solution of the director angle equation

The nonlinear elliptic equation (1.1b) is written as

$$-\nabla^2\theta = N(u,\theta) \tag{3.1}$$

where N is given by

$$N(u,\theta) = -\frac{q}{\nu}\sin(2\theta) + \frac{2}{\nu}|u|^2\cos(2\theta). \tag{3.2}$$

We see that $N(u,.)$ is decreasing on the interval $[0,\pi/4]$ and verifies

$$N(u,\theta) \le -\frac{4q}{\nu\,\pi}\theta + \frac{2}{\nu}|u|^2, \tag{3.3}$$

for all $\theta \in [0,\pi/4]$, and $u \in \mathbb{C}$. The function $N(u(x),\theta)$, $x \in \mathbb{R}^2$, satisfies conditions (1)–(2) and, since $|N(u,\theta)| \le C(|\theta| + |u|^2)$, N defines a Nemytskii operator for any $u \in L^4(\mathbb{R}^2)$. In the remainder of this section we will assume that $u \in L^4(\mathbb{R}^2)$ and we will write $N(u(x),\theta) = N(x,\theta)$ when no confusion can arise.

Lemma 3.1. *Given $u \in L^4(\mathbb{R}^2)$, equation* (1.1b) *has at most one solution $\theta \in H^2(\mathbb{R}^2)$ satisfying $0 \le \theta(x) \le \pi/4$ for all $x \in \mathbb{R}^2$.*

Proof. Let $\theta_1, \theta_2 \in H^2(\mathbb{R}^2)$ be solutions of (1.1b) taking values in the interval $[0,\pi/4]$. These solutions are also in $C^0(\mathbb{R}^2)$ by the Sobolev inequalities. By (3.2) their difference satisfies

$$-\nabla^2(\theta_1-\theta_2) = N(x,\theta_1) - N(x,\theta_2), \tag{3.4}$$

and since $N(x,.)$ is decreasing in $[0,\pi/4]$, we also have

$$(N(x,\theta_1) - N(x,\theta_2))(\theta_1-\theta_2)^+ \le 0,$$

a.e. in $\mathbb{R}^2$. Multiplying (3.4) by $(\theta_1-\theta_2)^+$ and integrating we therefore have

$$\left\|\nabla(\theta_1-\theta_2)^+\right\|_{L^2}^2 \le 0.$$

Interchanging θ_1 and θ_2, we similarly have $\left\|\nabla(\theta_2-\theta_1)^+\right\|_{L^2}^2 \le 0$. From the decomposition $\nabla(\theta_1-\theta_2) = \nabla(\theta_1-\theta_2)^+ - \nabla(\theta_2-\theta_1)^+$, it follows that $\nabla(\theta_1-\theta_2) = 0$ a.e. in $\mathbb{R}^2$. Since θ_1, θ_2 are continuous and decay at infinity, we obtain $\theta_1 \equiv \theta_2$. □

Lemma 3.2. *Consider $u \in L^4(\mathbb{R}^2)$ and let $\theta \in H^2(\mathbb{R}^2)$ be a corresponding solution of* (1.1b) *that also satisfies $-\pi/4 \leq \theta(x) \leq \pi/2$, for all $x \in \mathbb{R}^2$. Then $0 \leq \theta(x) \leq \pi/4$, for all $x \in \mathbb{R}^2$.*

Proof. By (3.2), $-\pi/4 \leq \theta \leq 0$ implies $N(x, \theta) \geq 0$, therefore $N(x, \theta)\theta^- \geq 0$ a.e. in $\mathbb{R}^2$. Multiplying (3.1) by θ^-, integrating and using $\nabla\theta \cdot \nabla\theta^- = -\left|\nabla\theta^-\right|^2$, we get

$$-\int_{\mathbb{R}^2} \left|\nabla\theta^-\right|^2 dx = \int_{\mathbb{R}^2} N(x, \theta)\theta^- \, dx \geq 0.$$

It follows that $\theta^- \equiv 0$.

For $\theta \in [\pi/4, \pi/2]$, we have $N(u, \theta) \leq 0$ and therefore $N(x, \theta) (\theta - \pi/4)^+ \leq 0$ a.e. in $\mathbb{R}^2$. Multiplying (3.1) by $(\theta - \pi/4)^+$, integrating and using $\nabla (\theta - \pi/4) \cdot \nabla (\theta - \pi/4)^+ = \left|\nabla (\theta - \pi/4)^+\right|^2$, we similarly obtain that $(\theta - \pi/4)^+$ is a constant and therefore $(\theta - \pi/4)^+ \equiv 0$. □

Corollary 3.1. *Let θ be as in Lemma 3.2, with $u \in L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then $0 \leq \theta(x) \leq \theta_{\max} < \pi/4$, for all $x \in \mathbb{R}^2$, where*

$$\theta_{\max} = \frac{1}{2} \arctan(2 \left\|u\right\|^2_{L^\infty} / q). \tag{3.5}$$

Proof. Consider $u \in L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Assume θ is a solution of (3.1), and define $\theta_{\max}$ as in (3.5). We need to show that $\theta(x) \leq \theta_{\max}$, for all $x \in \mathbb{R}^2$. By (3.2), $N(u(x), \theta(x)) \leq 0$ is equivalent to $1/2 \arctan(2|u(x)|^2/q) \leq \theta(x)$. Thus, if $\theta_{\max} \leq \theta(x) \leq \pi/4$ then $N(u(x), \theta(x)) \leq 0$. Therefore $N(x, \theta) (\theta - \theta_{\max})^+ \leq 0$ a.e. in $\mathbb{R}^2$. Multiplying (3.1) by $(\theta - \theta_{\max})^+$ and arguing as in Lemma 3.2, we see that $(\theta - \theta_{\max})^+ \equiv 0$. □

Lemma 3.3. *There exists a constant $C_{q,\nu} > 0$ such that if $\theta \in H^2(\mathbb{R}^2)$ is a solution of* (1.1b) *and satisfies $0 \leq \theta(x) \leq \pi/4$, for all $x \in \mathbb{R}^2$, then $\left\|\theta\right\|_{H^2} \leq C_{q,\nu} \left\|u\right\|^2_{L^4}$.*

Proof. Multiplying (1.1b) by θ, integrating, and using the assumption $\theta \in [0, \pi/4]$ we obtain

$$\left\|\nabla\theta\right\|^2_{L^2} \leq -\frac{4q}{\nu\pi} \left\|\theta\right\|^2_{L^2} + \frac{2}{\nu} \int_{\mathbb{R}^2} \left|u\right|^2 \theta \, dx.$$

Using Hölder's inequality, this implies

$$\left\|\nabla\theta\right\|^2_{L^2} + \frac{2q}{\nu\pi} \left\|\theta\right\|^2_{L^2} \leq \frac{\pi}{2\nu q} \left\|u\right\|^4_{L^4}. \tag{3.6}$$

By (3.1), (3.2) we also have

$$\left\|\nabla^2\theta\right\|_{L^2} \leq \frac{4q}{\nu\pi}\left\|\theta\right\|_{L^2} + \frac{2}{\nu}\left\|u\right\|_{L^4}^2 \leq \frac{4}{\nu}\left\|u\right\|_{L^4}^2 . \tag{3.7}$$

The lemma follows from (3.6) and (3.7). ∎

In order to solve (1.1b) we use the following definition. Let X be a Banach space, and consider a map $F : X \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ that satisfies $F(u,0) = 0$ and is continuous in a neighborhood of $(u,0)$. Then we will consider the property that for any $\varepsilon > 0$, there exists a neighborhood $\mathscr{V} \subset X \times H^2(\mathbb{R}^2)$ of $(u,0)$ for which

$$\left\|F(w,\psi_1) - F(w,\psi_2)\right\|_{L^2} \leq \varepsilon \left\|\psi_1 - \psi_2\right\|_{H^2} , \tag{3.8}$$

for all $(w,\psi_1), (w,\psi_2) \in \mathscr{V}$.

Property (3.8) combines Lipschitz continuity and superlinearity for the second component of F near $(u,0)$. In Lemma 3.4 we will see that (3.8) implies the existence of a unique solution $\psi(w)$ of $-\nabla^2\psi + V\psi = F(w,\psi)$ near $(u,0)$, V as in Lemma 2.2. Continuity of F in the first component makes ψ continuous in w. This setup will be then used to solve (1.1b).

Lemma 3.4. *Let X be a Banach space and consider a map $F : X \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. Assume that V satisfies the conditions of Lemma 2.2 and that F is continuous in a neighborhood of $(u,0)$ and satisfies (3.8) at $(u,0)$. Then there exists a neighborhood $\mathscr{U} \subset X$ of u and $\delta > 0$ such that for any $w \in \mathscr{U}$ the equation $-\nabla^2\psi + V\psi = F(w,\psi)$ has a unique solution $\psi \in H^2(\mathbb{R}^2)$ with $\left\|\psi\right\|_{H^2} < \delta$. Furthermore, the map $w \mapsto \psi$ from X to $H^2(\mathbb{R}^2)$ is continuous in $\mathscr{U}$.*

Proof. Let $K > 0$ be the constant of Lemma 2.2, and let $\mathscr{V}$ be a neighborhood of $(u,0)$ for which F satisfies (3.8) for any ϵ satisfying $0 < \varepsilon < 1/(2K)$. Then we can choose $\delta, r > 0$ such that $\overline{B}_X(u,\delta) \times \overline{B}_{H^2}(0,r) \subset \mathscr{V}$ and $\left\|F(w,0)\right\|_{L^2} < r/(2K)$ if $\left\|w-u\right\|_X \leq \delta$. Furthermore, for any $\left\|\psi\right\|_{H^2} \leq r$ we have

$$\begin{aligned} \left\|F(w,\psi)\right\|_{L^2} &\leq \left\|F(w,0)\right\|_{L^2} + \left\|F(w,\psi) - F(w,0)\right\|_{L^2} \\ &\leq \frac{r}{2K} + \varepsilon\left\|\psi\right\|_{H^2} \leq \frac{r}{K}. \end{aligned} \tag{3.9}$$

For $w \in X$, we define the map $\Gamma_w : H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$ given by $\Gamma_w(\psi) = \varphi$, where $\varphi \in H^2(\mathbb{R}^2)$ is the solution of $-\nabla^2\varphi + V\varphi = F(w,\psi)$. Therefore, Lemma 2.2, and (3.9) imply that if $\left\|w-u\right\|_X \leq \delta$ and $\left\|\psi\right\|_{H^2} \leq r$ then

$$\left\|\Gamma_w(\psi)\right\|_{H^2} = \left\|\varphi\right\|_{H^2} \leq K\left\|F(w,\psi)\right\|_{L^2} \leq r,$$

and therefore $\Gamma_w(\overline{B}_{H^2}(0,r)) \subset \overline{B}_{H^2}(0,r)$. Also, by (3.8) if $\psi_1, \psi_2 \in \overline{B}_{H^2}(0,r)$, then

$$\|F(w,\psi_1) - F(w,\psi_2)\|_{L^2} \le \varepsilon \|\psi_1 - \psi_2\|_{H^2},$$

therefore

$$\begin{aligned}\|\Gamma_w(\psi_1) - \Gamma_w(\psi_2)\|_{H^2} &\le K \|F(w,\psi_1) - F(w,\psi_2)\|_{L^2} \\ &\le K\varepsilon \|\psi_1 - \psi_2\|_{H^2} \le \frac{1}{2} \|\psi_1 - \psi_2\|_{H^2}.\end{aligned}$$

Thus Γ_w is a contraction mapping in $\overline{B}_{H^2}(0,r)$ and it admits a unique fixed-point ψ, i.e. there exists a unique $\psi \in \overline{B}_{H^2}(0,r)$ satisfying $-\nabla^2\psi + V\psi = F(w,\psi)$.

Let $w_0, w \in \overline{B}_X(u,\delta)$, $\psi_0 = \Gamma_{w_0}(\psi_0)$ and $\psi = \Gamma_w(\psi)$, we can write

$$\begin{aligned}-\nabla^2(\psi - \psi_0) + V(\psi - \psi_0) &= F(w,\psi) - F(w,\psi_0) \\ &\quad + F(w,\psi_0) - F(w_0,\psi_0),\end{aligned}$$

then $\|\psi - \psi_0\|_{H^2} \le K\varepsilon \|\psi - \psi_0\|_{H^2} + K \|F(w,\psi_0) - F(w_0,\psi_0)\|_{L^2}$. Thus,

$$\|\psi - \psi_0\|_{H^2} \le 2K \|F(w,\psi_0) - F(w_0,\psi_0)\|_{L^2}$$

and the continuity of $w \mapsto \psi$ follows from the continuity of $F(\cdot,\psi_0)$ near u. □

The existence of solutions (1.1b) is shown in Proposition 3.1 below, using a continuation idea and the setup of Lemma 3.4. We first show technical Lemmas 3.5, 3.6 on property (3.8) for the nonlinear terms.

Remark 3.1. Suppose $F_1, F_2 : X \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ satisfy (3.8) at u. Then for A_1, A_2 bounded operators in $L^2(\mathbb{R}^2)$, the map $F = A_1F_1 + A_2F_2 : X \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is continuous in a neighborhood of $(u,0)$ and satisfies property (3.8) at $(u,0)$.

Lemma 3.5. *Let $\alpha \in C^1(\mathbb{R})$, $u \in L^4(\mathbb{R}^2)$ and define $F : L^4(\mathbb{R}^2) \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by $F(w,\psi) = \left(|w|^2 - |u|^2\right)\alpha(\psi)$. Then F is continuous in a neighborhood of $(u,0)$ and satisfies property (3.8) at $(u,0)$.*

Proof. By $|F(w_1,\psi) - F(w_2,\psi)| \le (|w_1| + |w_2|)\,|w_1 - w_2|\,|\alpha(\psi)|$, we have

(3.10)
$$\|F(w_1,\psi) - F(w_2,\psi)\|_{L^2} \le (\|w_1\|_{L^4} + \|w_2\|_{L^4}) \|w_1 - w_2\|_{L^4} \|\alpha(\psi)\|_{L^\infty}.$$

Also, since α is continuously differentiable, there exists a constant $L > 0$ such that if $|\psi_1|, |\psi_2| \leq R$, then $|\alpha(\psi_1) - \alpha(\psi_2)| \leq L|\psi_1 - \psi_2|$. Thus, from the inequality $\|\psi\|_{L^\infty} \leq C\|\psi\|_{H^2}$, we have

(3.11)
$$\begin{aligned}\|F(w,\psi_1) - F(w,\psi_2)\|_{L^2} &\leq CL\left(\|u\|_{L^4} + \|w\|_{L^4}\right)\|w-u\|_{L^4}\|\psi_1-\psi_2\|_{H^2}\\ &\leq CL\left(2\|u\|_{L^4} + \delta\right)\delta\|\psi_1 - \psi_2\|_{H^2},\end{aligned}$$

assuming $\|w-u\|_{L^4} < \delta$, and $\|\psi_j\|_{H^2} \leq R/C$. Taking δ small enough, the statement follows from (3.10), (3.11). □

Lemma 3.6. *Let $u \in L^4(\mathbb{R}^2)$ and define $F : L^4(\mathbb{R}^2) \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by $F(w,\psi) = G(\psi)$, and $G(\psi) = (h_1 + h_2|u|^2)\beta(\psi)$, where $h_1, h_2 \in L^\infty(\mathbb{R}^2)$, and $\beta \in C^1(\mathbb{R})$ with $\beta(0) = 0$, $\beta'(0) = 0$. Then $F(w,\psi) = G(\psi)$ is continuous in a neighborhood of $(u, 0)$ and satisfies* (3.8) *at $(u, 0)$.*

Proof. Properties $\beta(0) = 0$, $\beta'(0) = 0$ imply that for any $\eta > 0$ there exists $R > 0$ such that $|\beta(\psi_1) - \beta(\psi_2)| \leq \eta|\psi_1 - \psi_2|$ for $|\psi_1|, |\psi_2| \leq R$. Then

$$|G(\psi_1) - G(\psi_2)| \leq \eta\left(|h_1| + |h_2||u|^2\right)|\psi_1 - \psi_2|,$$

and therefore

$$\|G(\psi_1) - G(\psi_2)\|_{L^2} \leq \eta\left(\|h_1\|_{L^\infty} + \|h_2\|_{L^\infty}\|u\|_{L^4}^2\right)\|\psi_1 - \psi_2\|_{H^2}. \tag{3.12}$$

This shows property (3.8) at $(u, 0)$. The map $F(w,\psi) = G(\psi)$ is independent of w and the statement follows. □

Proposition 3.1. *Let $u \in L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then there exists a unique solution $\theta \in H^2(\mathbb{R}^2)$ of* (1.1b) *satisfying $0 \leq \theta(x) \leq \pi/4$, for all $x \in \mathbb{R}^2$. Furthermore $\|\theta\|_{H^2} \leq C\|u\|_{L^4}^2$.*

Proof. Let $\mathscr{U} \subset L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be the set of functions u for which there exists a solution $\theta \in H^2(\mathbb{R}^2)$ of (1.1b), with the property that $0 \leq \theta \leq \pi/4$ everywhere in $\mathbb{R}^2$. We will prove that $\mathscr{U}$ is a nonempty open and closed subset of $L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Uniqueness and the bound on $\|\theta\|_{H^2}$ would then follow from Lemmas 3.1, 3.3 respectively.

The set $\mathscr{U}$ is nonempty since $u = 0 \in \mathscr{U}$, with $\theta = 0$. We will prove that $\mathscr{U}$ is closed. Let $\{u_n\}_{\mathbb{Z}^+}\} \in \mathscr{U}$ be a sequence that converges to u in $L^4(\mathbb{R}^2)$. By Lemma 3.3 we see that the corresponding sequence of solutions $\{\theta_n\}_{\mathbb{Z}^+}$ of (1.1b) is bounded in $H^2(\mathbb{R}^2)$. Then there exists $\theta \in H^2(\mathbb{R}^2)$ and a subsequence that converges weakly to θ in $H^2(\mathbb{R}^2)$ and therefore converges uniformly on compact subsets. Thus for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ we have

$\lim_{n\to\infty}\langle\nabla^2\theta_n,\varphi\rangle = \langle\nabla^2\theta,\varphi\rangle$ and that $\sin(2\theta_n)\varphi, \cos(2\theta_n)\varphi$ converge uniformly to $\sin(2\theta)\,\varphi, \cos(2\theta)\,\varphi$ respectively. It follows that θ is a solution of (1.1b) corresponding to u. Since the θ_n converge pointwise to θ, we have $0 \le \theta \le \pi/4$, which implies $\mathscr{U}$ is closed.

To see that $\mathscr{U}$ is open, it is enough to consider $u \in \mathscr{U}$ and the corresponding solution θ of (1.1b), and prove that there exists $\delta > 0$ such that $u+v \in \mathscr{U}$ if $v \in L^4(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ with $\|v\|_{L^4\cap L^\infty} < \delta$. We write the solution for $u+v$ as $\theta+\psi$, then ψ satisfies $-\nabla^2\psi + V\psi = F(u+v,\psi)$, where V is the potential given by $V = \frac{2q}{\nu}\cos(2\theta) + \frac{4}{\nu}|u|^2\sin(2\theta)$, and

$$\begin{aligned} F(w,\psi) = {} & \frac{2}{\nu}\left(|w|^2 - |u|^2\right)\cos(2(\theta+\psi)) \\ & + \left(\frac{q\sin(2\theta)}{\nu} - \frac{2}{\nu}|u|^2\cos(2\theta)\right)(1-\cos(2\psi)) \\ & + \left(\frac{q\cos(2\theta)}{\nu} + \frac{2}{\nu}|u|^2\sin(2\theta)\right)(2\psi - \sin(2\psi)). \end{aligned}$$

We can see that $V \in L^\infty(\mathbb{R}^2)$ and, since $0 \le \theta \le \pi/4$, $V \ge 0$. As $\theta \in H^2(\mathbb{R}^2)$, we also have $\lim_{|x|\to\infty}\theta(x) = 0$, hence

$$\liminf_{|x|\to\infty} V(x) \ge \frac{2q}{\nu}.$$

Therefore V verifies the conditions of Lemma 2.2. By Lemmas 3.5, 3.6 and Remark 3.1, we see that F is continuous from $L^4(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)\times H^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ and verifies (3.8) at $(u,0)$. Using Lemma 3.4, there exists $r>0$ such that if $\|v\|_{L^4} < r$ then $-\nabla^2\psi + V\psi = F(u+v,\psi)$ has a unique solution $\psi \in H^2(\mathbb{R}^2)$ with $\|\psi\|_{H^2} \le \delta$. Taking $r>0$ small enough, we can assume $|\psi| < \pi/4$ and then $-\pi/4 < \theta+\psi < \pi/2$ for all $x \in \mathbb{R}^2$. Then Lemma 3.2 implies that $0 \le \theta+\psi \le \pi/4$ everywhere $\mathbb{R}^2$. Thus $\mathscr{U}$ is open. Since $\mathscr{U}$ closed, open and nonempty, we conclude $\mathscr{U} = L^4(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. □

Lemma 3.1 shows the existence of a global continuous branch of solutions $\theta(u) \in H^2(\mathbb{R}^2)$ of(1.1b), defined for all $u \in \mathscr{U} = L^4(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. The unique continuation arguments above, i.e. Lemmas 3.4, 3.2, imply that any other continuous branch of solutions $\tilde\theta(u) \in H^2(\mathbb{R}^2)$, $u \in \tilde{\mathscr{U}} \subset L^4(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, that intersects the branch of Lemma 3.1 must coincide with it. In particular a branch that includes the origin in its domain must coincide with the one described by the lemma above.

4 Well–posedness of the evolution problem

We now consider the initial value problem for system (1.1), written as

$$u(z) = W(z)u_0 + \mathrm{i}\int_0^z W(z-z')u(z')\sin(2\theta(z'))dz', \tag{4.1a}$$

$$-\nabla^2\theta = N(u,\theta), \tag{4.1b}$$

where $N(u,\theta) = -\frac{q}{\nu}\sin(2\theta) + \frac{2}{\nu}|u|^2\cos(2\theta)$, and W is the unitary group generated by the operator $\frac{\mathrm{i}}{2}\nabla^2$.

Proposition 4.1. *The map $\Theta : H^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$ defined by $\Theta(u) = \theta$, where θ is the solution of* (4) *satisfies*

$$\begin{aligned}\|\Theta(u_1)-\Theta(u_2)\|_{H^2} \le\, & C_{\nu,q}\left(\|u_1\|_{H^1}, \|u_2\|_{H^1}\right)\left(1+\|u_1\|_{L^\infty}^2+\|u_2\|_{L^\infty}^2\right)\\ & \times \|u_1-u_2\|_{H^1},\end{aligned} \tag{4.2}$$

and is therefore locally Lipschitz continuous.

Proof. Let $u_1, u_2 \in H^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$, with $R = \max\{\|u_1\|_{H^1\cap L^\infty}, \|u_2\|_{H^1\cap L^\infty}\}$, and let $\theta_1, \theta_2 \in H^2(\mathbb{R}^2)$ be their respective solutions of , as in Proposition (3.1). By corollary 3.1, we see that $0 \le \theta_j(x) \le \theta_{\max}$, for all $x\in\mathbb{R}^2$, where $\theta_{\max} = \frac{1}{2}\arctan(2R^2/q) \in [0,\pi/4]$. We then have

$$|\sin(2\theta_1)-\sin(2\theta_2)| \ge \frac{2q}{\sqrt{4R^4+q^2}}\,|\theta_1-\theta_2|\,, \tag{4.3}$$

by the mean value theorem. Also, the difference between two solutions of (4.1a) satisfies

(4.4)

$$\begin{aligned}-\nabla^2(\theta_1-\theta_2) = &-\frac{q}{\nu}\left(\sin(2\theta_1)-\sin(2\theta_2)\right)\\ &+\frac{2}{\nu}|u_1|^2\left(\cos(2\theta_1)-\cos(2\theta_2)\right)+\frac{2}{\nu}\left(|u_1|^2-|u_2|^2\right)\cos(2\theta_2),\end{aligned}$$

so that multiplying by $\theta_1-\theta_2$ and integrating by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla(\theta_1-\theta_2)|^2\,dx = &-\frac{q}{\nu}\int_{\mathbb{R}^2}\left(\sin(2\theta_1)-\sin(2\theta_2)\right)(\theta_1-\theta_2)\,dx\\ &+\frac{2}{\nu}\int_{\mathbb{R}^2}|u_1|^2\left(\cos(2\theta_1)-\cos(2\theta_2)\right)(\theta_1-\theta_2)\,dx\\ &+\frac{2}{\nu}\int_{\mathbb{R}^2}\left(|u_1|^2-|u_2|^2\right)\cos(2\theta_2)\,(\theta_1-\theta_2)\,dx.\end{aligned} \tag{4.5}$$

To estimate the right-hand side of (4.5), we use (4.3), and $\theta_j \in [0, \pi/4]$ to see that

$$(\sin(2\theta_1) - \sin(2\theta_2))(\theta_1 - \theta_2) \geq 2q\left(4R^4 + q^2\right)^{-1/2}(\theta_1 - \theta_2)^2,$$

thus

$$-\frac{q}{\nu}\int_{\mathbb{R}^2}(\sin(2\theta_1) - \sin(2\theta_2))(\theta_1 - \theta_2)\,dx \leq -\frac{2q^2}{\nu\sqrt{4R^4 + q^2}}\left\|(\theta_1 - \theta_2)\right\|_{L^2}^2. \tag{4.6}$$

Also, since $\cos(2\theta)$ is decreasing in the interval $[0, \pi/4]$, we see that

$$\int_{\mathbb{R}^2}|u_1|^2(\cos(2\theta_1) - \cos(2\theta_2))(\theta_1 - \theta_2)\,dx \leq 0. \tag{4.7}$$

To estimate the third integral in (4.5) we use Hölder's inequality to see that

$$\begin{aligned}\int_{\mathbb{R}^2}\left(|u_1|^2 - |u_2|^2\right)\cos(2\theta_2)(\theta_1 - \theta_2)\,dx \leq \left(\|u_1\|_{L^4} + \|u_2\|_{L^4}\right)\|u_1 - u_2\|_{L^4}\\ \times\|\theta_1 - \theta_2\|_{L^2}.\end{aligned} \tag{4.8}$$

Letting $a = q^2/(\nu\sqrt{4R^4 + q^2})$, and using (4.5)-(4.8) and the Gagliardo-Nirenberg inequality (2.6) we have

$$\|\nabla(\theta_1 - \theta_2)\|_{L^2}^2 + a\|(\theta_1 - \theta_2)\|_{L^2}^2 \leq \frac{1}{\nu^2 a}C\left(\|u_1\|_{H^1} + \|u_2\|_{H^1}\right)^2\|u_1 - u_2\|_{L^4}^2,$$

and therefore

$$\|\nabla(\theta_1 - \theta_2)\|_{L^2}^2 \leq \frac{\sqrt{4R^4 + q^2}}{\nu q^2}C\left(\|u_1\|_{H^1} + \|u_2\|_{H^1}\right)^2\|u_1 - u_2\|_{L^4}^2, \tag{4.9}$$

$$\|(\theta_1 - \theta_2)\|_{L^2}^2 \leq \frac{4R^4 + q^2}{q^4}C\left(\|u_1\|_{H^1} + \|u_2\|_{H^1}\right)^2\|u_1 - u_2\|_{L^4}^2. \tag{4.10}$$

Using the above inequalities, and considering $C_{\nu,q}$ that satisfies

$$\frac{\sqrt{4R^4 + q^2}}{\nu q^2} + \frac{4R^4 + q^2}{q^4} \leq C_{\nu,q}(1 + R^2)^2$$

we obtain

$$\|\theta_1 - \theta_2\|_{H^1} \leq C_{\nu,q}(1 + R^2)\left(\|u_1\|_{H^1} + \|u_2\|_{H^1}\right)\|u_1 - u_2\|_{L^4}, \tag{4.11}$$

from which we have the final estimate for the H^1 norm

$$\begin{aligned}\|\theta_1-\theta_2\|_{H^1} \le\, & C_{\nu,q}\left(\|u_1\|_{H^1}+\|u_2\|_{H^1}\right)\left(1+\|u_1\|_{L^\infty}^2+\|u_2\|_{L^\infty}^2\right)\\ & \times\|u_1-u_2\|_{L^4}\,.\end{aligned} \tag{4.12}$$

To obtain a Lipschitz estimate for $\|\theta_1-\theta_2\|_{H^2}$ we will use equation (4) for the θ_j, and (4.3) to get

$$\begin{aligned}\left|\nabla^2(\theta_1-\theta_2)\right| \le\, & \frac{2q}{\nu}\left|\theta_1-\theta_2\right|+\frac{4}{\nu}\left|u_1\right|^2\left|\theta_1-\theta_2\right|\\ & +\frac{2}{\nu}(|u_1|+|u_2|)\left|u_1-u_2\right|.\end{aligned} \tag{4.13}$$

Using (4.10) it then follows that

$$\begin{aligned}\left\|\nabla^2(\theta_1-\theta_2)\right\|_{L^2} \le\, & C_{\nu,q}\left(\|u_1\|_{H^1}+\|u_2\|_{H^1}\right)\left(1+\|u_1\|_{L^\infty}^2+\|u_2\|_{L^\infty}^2\right)\\ & \times\|u_1-u_2\|_{L^4}\,.\end{aligned} \tag{4.14}$$

Combining the above inequality, (4.12) and Gagliardo-Nirenberg (2.6), we obtain the estimate (4.2). □

Lemma 4.1. *Let $f\in L^1([0,\zeta],H^1(\mathbb{R}^2))$, and define g by*

$$g(z)=\mathrm{i}\int_0^z W(z-z')f(z')dz'.$$

Then $g\in Y_\zeta$ and satisfies $\|g\|_{Y_\zeta}\le C_{1,2}\,\|f\|_{L^1([0,\zeta],H^1)}$.

Proof. Since $W(z)$ is a unitary operator, we have $\|g\|_{C([0,\zeta],H^1)}\le\|f\|_{L^1([0,\zeta],H^1(\mathbb{R}^2))}$. Using

$$\nabla g(z)=\mathrm{i}\int_0^z W(z-z')\nabla f(z')dz',$$

and the second Strichartz estimate (2.2) with $p=q=4$, $r=1$, $\gamma=1$, we have $\|\nabla g\|_{L^4([0,\zeta],L^4)}\le C_{1,2}\,\|\nabla f\|_{L^1([0,\zeta],L^2)}$. The statement then follows immediately from the definition of Y_ζ. □

Lemma 4.2. *Let $u_0\in H^1(\mathbb{R}^2)$, and $h(z)=W(z)u_0$. Then $\|h\|_{Y_\zeta}\le C_4\,\|u_0\|_{H^1(\mathbb{R}^2)}$.*

Proof. The statement follows from the first Strichartz estimate (2.2) with $p=q=4$, and the fact that $z\mapsto h(z)\in C([0,\zeta];H^1)$. □

Lemma 4.3. *Let B be the map defined by $B(u)=u\sin(2\Theta(u))$. Then B is bounded from Y_ζ to $L^1([0,\zeta],H^1(\mathbb{R}^2))$, moreover for any $R>0$ there exists $C>0$ such that $u\in Y_\zeta$ and $\|u\|_{Y_\zeta}\le R$ imply $\|B(u)\|_{L^1([0,\zeta],H^1(\mathbb{R}^2))}\le C\,\zeta\,\|u\|_{Y_\zeta}$.*

Proof. Let $\theta = \Theta(u)$. By $|\sin(2\theta)| \leq 1$, therefore $\|B(u)\|_{L^2} \leq \|u\|_{L^2}$. Using $\nabla B(u) = \nabla u \sin(2\theta) + 2u\cos(2\theta)\nabla\theta$, Lemma 3.3, and Gagliardo-Nirenberg (2.6), we have

$$\begin{aligned} \|\nabla B(u)\|_{L^2} &\leq \|\nabla u\|_{L^2} + 2\|u\|_{L^4}\|\nabla\theta\|_{L^4} \\ &\leq \|\nabla u\|_{L^2} + C\|u\|_{L^4}\|\theta\|_{H^2} \\ &\leq \tilde{C}(\|\nabla u\|_{L^2} + \|u\|_{L^4}^3) \leq \tilde{C}(\|u\|_{H^1} + \|u\|_{H^1}^3). \end{aligned} \tag{4.15}$$

The result follows by integration over $[0, \zeta]$. □

Lemma 4.4. *The map $B : Y_\zeta \to L^1([0,\zeta], H^1(\mathbb{R}^2))$ defined in Lemma 4.3 is locally Lipschitz, i.e. for any $R > 0$ there exists $C > 0$ such that $u_1, u_2 \in Y_\zeta$ and $\|u_1\|_{Y_\zeta}, \|u_2\|_{Y_\zeta} \leq R$ imply*

$$\|B(u_1) - B(u_2)\|_{L^1([0,\zeta],H^1(\mathbb{R}^2))} \leq C\,(\zeta + \zeta^{2/3})\,\|u_1 - u_2\|_{C([0,\zeta],H^1)}. \tag{4.16}$$

Proof. Let $u_1, u_2 \in Y_\zeta$ with $\|u_1\|_{Y_\zeta}, \|u_2\|_{Y_\zeta} \leq R$, we can see that

$$|B(u_1) - B(u_2)| \leq |u_1 - u_2| + 2|u_2|\,|\theta_1 - \theta_2|,$$

with $\theta_j = \Theta(u_j)$. Therefore

$$\begin{aligned} \|B(u_1) - B(u_2)\|_{L^2} &\leq C(\|u_1 - u_2\|_{L^2} + \|u_2\|_{L^4}\|\theta_1 - \theta_2\|_{L^4}) \\ &\leq C(\|u_1 - u_2\|_{H^1} + \|u_2\|_{H^1}\|\theta_1 - \theta_2\|_{H^1}). \end{aligned} \tag{4.17}$$

On the other hand, it is easy to see that

$$\begin{aligned} |\nabla B(u_1) - \nabla B(u_2)| \leq{}& |\nabla(u_1 - u_2)| + 2|\nabla u_2|\,|\theta_1 - \theta_2| \\ &+ 2|u_1 - u_2|\,|\nabla\theta_1| + 2|u_2|\,|\nabla\theta_2 - \nabla\theta_1| \\ &+ 4|u_2|\,|\theta_2 - \theta_1|\,|\nabla\theta_2| = I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{4.18}$$

From the embeddings $H^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ and $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, we estimate each term as

$$\begin{aligned} &\|I_1\|_{L^2} \leq \|\nabla(u_1 - u_2)\|_{L^2} \leq \|u_1 - u_2\|_{H^1}, \\ &\|I_2\|_{L^2} \leq C\|\nabla u_2\|_{L^2}\|\theta_1 - \theta_2\|_{L^\infty} \leq C\|u_2\|_{H^1}\|\theta_1 - \theta_2\|_{H^2}, \\ &\|I_3\|_{L^2} \leq C\|\nabla\theta_1\|_{L^4}\|u_1 - u_2\|_{L^4} \leq C\|\theta_1\|_{H^2}\|u_1 - u_2\|_{H^1}, \\ &\|I_4\|_{L^2} \leq C\|u_2\|_{L^4}\|\nabla\theta_2 - \nabla\theta_1\|_{L^4} \leq C\|u_2\|_{H^1}\|\theta_2 - \theta_1\|_{H^2}, \\ &\|I_5\|_{L^2} \leq C\|u_2\|_{L^4}\|\nabla\theta_2\|_{L^4}\|\theta_2 - \theta_1\|_{L^\infty} \leq C\|u_2\|_{H^1}\|\theta_2\|_{H^2}\|\theta_2 - \theta_1\|_{H^2}. \end{aligned} \tag{4.19}$$

From Lemma 3.3 we also have $\|\theta_j\|_{H^2} \leq C\|u_j\|_{H^1}^2$. Thus, Proposition 4.1, the Gagliardo–Nirenberg inequality (2.5), and (4.17)-(4.19) imply that

$$\|B(u_1) - B(u_2)\|_{H^1} \leq C(R)(1 + \|\nabla u_1\|_{L^4}^{4/3} + \|\nabla u_2\|_{L^4}^{4/3}) \|u_1 - u_2\|_{H^1}.$$

Integrating over $[0, \zeta]$, we obtain

$$\|B(u_1) - B(u_2)\|_{L^1([0,\zeta],H^1)} \leq C(R) \|u_1 - u_2\|_{C([0,\zeta],H^1)} \times \int_0^\zeta (1 + \|\nabla u_1\|_{L^4}^{4/3} + \|\nabla u_2\|_{L^4}^{4/3})\, dz,$$

and using Hölder's inequality we finally have

$$\|B(u_1) - B(u_2)\|_{L^1([0,\zeta],H^1)} \leq C\left(\zeta + \zeta^{2/3}\right) \|u_1 - u_2\|_{C([0,\zeta],H^1)}.$$

for some constant C that depends on R, as stated. □

The local existence of solutions of the evolution equation (4.1) is established by the next theorem.

Theorem 4.1. *Given $u_0 \in H^1(\mathbb{R}^2)$, there exists $\zeta = \zeta(\|u_0\|_{H^1}) > 0$ and a unique $(u, \theta) \in Y_\zeta \times L^\infty([0,\zeta], H^2(\mathbb{R}^2))$ that satisfies (4.1) and $\theta \in [0, \pi/4]$. Furthermore, the map $u_0 \mapsto u$ is continuous from $H^1(\mathbb{R}^2)$ to Y_ζ.*

Proof. From Lemmas 4.1, 4.2, and 4.3, the map Γ defined in Y_ζ by

$$(\Gamma u)(z) = W(z)\, u_0 + \mathrm{i} \int_0^z W(z - z') B(u(z'))\, dz', \quad z \in [0, \zeta], \tag{4.20}$$

satisfies $\Gamma u \in Y_\zeta$. (The dependence of Γ on u_0 is not made explicit in this notation.) By Lemma 3.3, we also have that $u \in Y_\zeta$ implies $\theta = \Theta(u) \in L^\infty([0, \zeta], H^2(\mathbb{R}^2))$. Define $h \in Y_\zeta$ by $h(z) = W(z)\, u_0$, $z \in [0, \zeta]$, and consider the closed ball $\overline{\mathcal{B}}_h(R) \subset Y_\zeta$ that is centered at h and has radius $R > 0$. Using Lemmas 4.1, 4.3 we see that if ζ is sufficiently small then

$$\|\Gamma u - h\|_{Y_\zeta} \leq C(\|h\|_{Y_\zeta} + R)\zeta \leq R. \tag{4.21}$$

Thus Γ maps the closed ball to its interior. To complete the argument we will prove that Γ is a contraction in $\overline{\mathcal{B}}_h(R)$. Then it will have a unique fixed point in $\overline{\mathcal{B}}_h(R)$.

Let u_1, $u_2 \in Y_\zeta$. Then for $0 \leq z \leq \zeta$

$$(\Gamma u_1)(z) - (\Gamma u_2)(z) = \mathrm{i} \int_0^z W(z - z')\left(B(u_1(z')) - B(u_2(z'))\right) dz', \tag{4.22}$$

and by Lemmas 4.1, 4.4 we have

$$\begin{aligned}\|\Gamma(u_1)-\Gamma(u_2)\|_{Y_\zeta} &\leq C\,\|B\,(u_1)-B\,(u_2)\|_{L^1([0,\zeta],H^1)}\\ &\leq C\,(R)\,\left(\zeta+\zeta^{2/3}\right)\|u_1-u_2\|_{Y_\zeta}\,.\end{aligned} \tag{4.23}$$

Thus taking ζ such that $C\,(R)\,(\zeta+\zeta^{2/3})<1$ we see that Γ is a contraction in $\overline{\mathcal{B}}_h(R)$.

To see the continuity on the initial conditions, we consider solutions u_j, with respective initial conditions v_j, $j=1,\,2$. We use the notation $\Gamma_{v_j}(u_j)$ for the map Γ of (4.20). By $u_j=\Gamma_{v_j}(u_j)$ we can immediately combine and Lemma (4.2) to see that for ζ sufficiently small we have $\|u_1-u_2\|_{Y_\zeta}\leq C\,\|v_1-v_2\|_{H^1}$, as required. $\square$

The above result of local existence, and the conservation of energy (1.4), leads to the following global existence statement.

Theorem 4.2 (Global existence)**.** *Given $u_0\in H^1(\mathbb{R}^2)$, there exists a unique $(u,\theta)\in C(\mathbb{R},H^1(\mathbb{R}^2))\times L^\infty(\mathbb{R},H^2(\mathbb{R}^2))$ such that $\theta\in[0,\pi/4]$ and $\nabla u\in L^4_{loc}(\mathbb{R},L^4(\mathbb{R}^2))$ solution of* (1.1)*.*

Proof. To obtain global existence we use smoother solutions (u,θ) and the continuous dependence. Given $u\in H^2(\mathbb{R}^2)$, we use the fact that $H^2(\mathbb{R}^2)$ is a Banach algebra, and the argument of Proposition 4.1, to prove that $\Theta(u)\in H^4(\mathbb{R}^2)$. Moreover, we use the argument of Theorem 4.1 to see that for $u_0\in H^2(\mathbb{R}^2)$ we have the local solution $(u,\theta)\in C([0,\zeta],H^2(\mathbb{R}^2))\times L^\infty([0,\zeta],H^4(\mathbb{R}))$, with $u\in C^1([0,\zeta],L^2(\mathbb{R}^2))$. Considering such (u,θ), we use the explicit form of the energy in (1.4), to see that

$$H(u,\theta)\geq\frac{1}{4}\,\|\nabla u\|^2_{L^2}-\frac{1}{2}\,\|u\|^2_{L^2}$$

, and therefore $\|u\|^2_{H^1}\leq 4H(u,\theta)+3\,\|u\|^2_{L^2}$, where the right side is a constant depending on $\|u_0\|_{H^1}$. From continuous dependence on initial data, we obtain an a priori bound for $\|u\|^2_{H^1}$. Now, by an usual prolongation argument we can assert that u is defined on $\mathbb{R}$. $\square$

We conclude this section showing that if $\|u_0\|_{L^2}$ is sufficiently small, then the solution of (4.1) satisfies $u\in L^4(\mathbb{R},L^4(\mathbb{R}^2))$. As a consequence the soliton solutions considered in the next section can not have arbitrarily small L^2-norm.

Proposition 4.2. *There exists $C>0$ such that if $(u,\theta)\in Y_\zeta\times L^\infty([0,\zeta],H^2)$ is the solution of* (4.1)*, then*

$$\|u\|_{L^4([0,\zeta],L^4)}\leq C\,\|u_0\|_{L^2}+C\,\|u\|^3_{L^4([0,\zeta],L^4)}\,. \tag{4.24}$$

Proof. The solution u of (4.1a) satisfies

$$u(z) = W(z)u_0 + \mathrm{i}\int_0^z W(z-z')u(z')\sin(2\theta(z'))dz' = h(z) + g(z).$$

From Strichartz estimates (2.2) we obtain

$$\begin{aligned} \|h\|_{L^4([0,\zeta],L^4)} &\leq C_{4,2}\,\|u_0\|_{L^2}\,, \\ \|g\|_{L^4([0,\zeta],L^4)} &\leq C_{4,4/3}\,\|u\sin(2\theta)\|_{L^{4/3}([0,\zeta],L^{4/3})}\,. \end{aligned} \tag{4.25}$$

Then Hölder's inequality, and $\|\theta\|_{L^2} \leq \tilde{C}\,\|u\|_{L^4}^2$ from Lemma 3.3 yield

$$\begin{aligned} \|u\sin(2\theta)\|_{L^{4/3}([0,\zeta],L^{4/3})}^{4/3} &\leq C_{4,4/3}^{4/3}\int_0^\zeta \|u(z)\|_{L^4}^{4/3}\,\|\theta(z)\|_{L^2}^{4/3}\,dz \\ &\leq \tilde{C}C_{4,4/3}^{4/3}\int_0^\zeta \|u(z)\|_{L^4}^4\,dz \leq C\,\|u\|_{L^4([0,\zeta],L^4)}^4\,. \end{aligned} \tag{4.26}$$

The statement follows immediately from (4.25), (4.26), with C depending on $C_{4,2}$, $C_{4,4/3}$, and $\tilde{C}$. ◻

Lemma 4.5. *Let $f(\tau) = a + C\,\tau^3$ with $0 < a < \frac{2}{3\sqrt{3C}}$, $C > 0$. Then there exist $0 < \tau_1 < \tau_2$ such that $\tau > f(\tau)$, $\tau > 0$ are satisfied if and only if $\tau \in (\tau_1, \tau_2)$.*

Proposition 4.3. *There exists $a_0 > 0$ such that if $u_0 \in H^1(\mathbb{R}^2)$ satisfies $\|u_0\|_{L^2} < a_0$, then the solution of (1.1) satisfies $\|u\|_{L^4(\mathbb{R},L^4)} < \infty$.*

Proof. Let $a_0 > 0$ such that $a_0 < \frac{2}{3\sqrt{3C}}$ and let $\tau(\zeta) = \|u\|_{L^4([0,\zeta],L^4)}$. It is easy to see $\tau(\zeta)$ is continuous and $\tau(0) = 0$. By Proposition 4.2, $\tau(\zeta) \leq f(\tau(\zeta))$ for any $\zeta > 0$, where f is as in Lemma 4.5. Lemma 4.5 then implies $\tau(\zeta) \leq \tau_1$, for all $\zeta > 0$. The result can be easily extended to negative ζ, e.g. using the complex conjugate of u_0 as initial condition. ◻

5 Existence of ground states

In this section we study the existence of solutions (u, θ) of the stationary problem associated to the system (1.1). Using the ansatz $u(x,z) = e^{\mathrm{i}\sigma z}v(x)$ with $\sigma \in \mathbb{R}$ and $\theta(x,z) = \phi(x)$, equations (1.1) become

$$\begin{aligned} 0 &= \nabla^2 v - 2\sigma v + 2v\sin(2\phi), \\ 0 &= \nabla^2\phi - \frac{q}{\nu}\sin(2\phi) + \frac{2}{\nu}\,|v|^2\cos(2\phi). \end{aligned} \tag{5.1}$$

Let

$$S_a = \left\{ (v, \phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) : \|v\|_{L^2}^2 = a \right\}, . \tag{5.2}$$

where $H(v, \phi)$ is given in (1.4). We will show the existence of an element (v, θ) of S_a attaining the infimum $J_a = \inf_{(v,\theta) \in S_a} H(v, \theta)$. We can see that H is differentiable in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, and that such a minimizer (v, θ) must satisfy (5.1) for some real σ.

We first observe that $H(v, \phi) \geq -\frac{a}{2}$, and therefore $J_a > -\infty$. Also we have $H(0, \phi), H(v, 0) \geq 0$.

Let $P(t)$ be a $\pi/2$–periodic continuous function defined on $[0, \pi/2]$ as follows: $P(t) = t$ if $0 \leq t \leq \pi/4$, and $P(t) = \pi/2 - t$ if $\pi/4 \leq t \leq \pi/2$.

Lemma 5.1. *Let $(v, \phi) \in S_a$. Then $(|v|, P(\phi)) \in S_a$, and*

$$H(|v|, P(\phi)) \leq H(v, \phi).$$

Proof. First, $\nabla P(\phi) = P'(\phi)\nabla\theta = \pm\nabla\theta$ implies $\|\nabla P(\phi)\|_{L^2} = \|\nabla\phi\|_{L^2}$. Also, $|\nabla |v|| \leq |\nabla v|$ implies $\|\nabla |v|\|_{L^2} \leq \|\nabla v\|_{L^2}$. Moreover, we check that $\sin(2P(\phi)) = |\sin(2\phi)|$, and $\cos(2P(\phi)) = |\cos(2\phi)|$, for all ϕ real. By (1.4) it then follows that $H(v, \phi) \geq H(|v|, P(\phi))$. □

By Lemma (5.1) we can restrict our attention to functions (v, ϕ) such that $v \geq 0$ and $0 \leq \phi \leq \pi/4$ almost everywhere in $\mathbb{R}^2$.

Let φ^* denote the symmetric decreasing rearrangement of $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ a measurable function such that $|\{x \in \mathbb{R}^n : \varphi(x) > t\}| < \infty$ for any $t > 0$. We recall the following lemma, see [10].

Lemma 5.2. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a increasing continuous function such that $f(0) = 0$, then for all $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ measurable, $(f \circ \varphi)^* = f \circ \varphi^*$*

Proposition 5.1. *Let $(v, \phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ with $v \geq 0$ and $0 \leq \phi \leq \pi/4$, a.e. in $\mathbb{R}^2$. Then $H(v^*, \phi^*) \leq H(v, \phi)$, where v^* and ϕ^* are the symmetric decreasing rearrangements of v and ϕ respectively.*

Proof. Applying the Pólya-Szegö inequality, we have that

$$\frac{1}{4}\|\nabla v^*\|_{L^2}^2 + \frac{\nu}{4}\|\nabla\phi^*\|_{L^2}^2 \leq \frac{1}{4}\|\nabla v\|_{L^2}^2 + \frac{\nu}{4}\|\nabla\phi\|_{L^2}^2 . \tag{5.3}$$

The functions $1 - \cos(2\phi)$, $\sin(2\phi)$ are increasing continuous on $[0, \pi/4]$ and vanish at the origin. Lemma 5.2 then implies $(1 - \cos(2\phi))^* = 1 - \cos(2\phi^*)$, and $(\sin(2\phi))^* = \sin(2\phi^*)$. The first equality yields

$$\int_{\mathbb{R}^2} \frac{q}{4}(1 - \cos(2\phi))\, dx = \int_{\mathbb{R}^2} \frac{q}{4}(1 - \cos(2\phi))^*\, dx = \int_{\mathbb{R}^2} \frac{q}{4}(1 - \cos(2\phi^*))\, dx, \tag{5.4}$$

while the second equality and the product rearrangement inequality of [10], ch.3.4, imply

$$\int_{\mathbb{R}^2} v^2 \sin(2\phi) dx \leq \int_{\mathbb{R}^2} v^{*2} \sin(2\phi)^* dx = \int_{\mathbb{R}^2} v^{*2} \sin(2\phi^*) dx. \tag{5.5}$$

The conclusion follows immediately from (5.3)-(5.5). □

Proposition 5.2. *There exists $\tilde{a} > 0$ such that if $0 < a \leq \tilde{a}$, then $J_a = 0$. Also, there exists $b > \tilde{a} > 0$ such that $J_a < 0$ for all $a \geq b$.*

Proof. Since $0 \leq \phi \leq \pi/4$ we have $1 - \cos(2\phi) \geq \phi^2$, and therefore

$$\int_{\mathbb{R}^2} \frac{q}{4} (1 - \cos(2\phi))\, dx \geq \frac{q}{4} \|\phi\|_{L^2}^2 .$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}^2} v^2 \sin(2\phi) dx &\leq \|v\|_{L^4}^2 \|\sin(2\phi)\|_{L^2} \\ &\leq 2 \|v\|_{L^4}^2 \|\phi\|_{L^2} \leq \frac{2}{q} \|v\|_{L^4}^4 + \frac{q}{2} \|\phi\|_{L^2}^2 . \end{aligned}$$

Thus

$$H(v,\phi) \geq \frac{1}{4} \|\nabla v\|_{L^2}^2 + \frac{\nu}{4} \|\nabla \phi\|_{L^2}^2 - \frac{1}{q} \|v\|_{L^4}^4 ,$$

so that by the Gagliardo-Nirenberg inequality (2.6), $\|v\|_{L^4}^4 \leq C\, a \|\nabla v\|_{L^2}^2$, we have $J_a \geq 0$ if $a \leq \tilde{a} = q/(4C)$.

Consider now some $v \in H^1(\mathbb{R}^2)$ with $\|v\|_{L^2}^2 = a$, and let $u_\lambda(x) = \lambda v(\lambda x)$. We have $\|u_\lambda\|_{L^2}^2 = a$ and $\|\nabla u_\lambda\|_{L^2}^2 = \lambda^2 \|\nabla u_\lambda\|_{L^2}^2$, therefore $H(u_\lambda, 0) = \lambda^2 H(v, 0) \to 0$ as $\lambda \to 0$. We conclude that $J_a = 0$ for $a \leq \tilde{a}$, proving the first statement.

To show the second statement, we use the fact that if $\phi \in [0, \pi/4]$ then $1 - \cos(2\phi) \leq 2\phi^2$, and $\sin(2\phi) \geq \frac{4}{\pi}\phi$. Letting $v = \alpha\phi$, $\alpha \in \mathbb{R}$, we then have

$$\begin{aligned} H(\alpha\phi, \phi) &\leq \frac{\alpha^2 + \nu}{4} \|\nabla \phi\|_{L^2}^2 + \frac{q}{2} \|\phi\|_{L^2}^2 - \frac{2\alpha^2}{\pi} \|\phi\|_{L^3}^3 \\ &= \frac{\alpha^2 + \nu}{4} \left(\|\nabla \phi\|_{L^2}^2 + \frac{2q}{\alpha^2 + \nu} \|\phi\|_{L^2}^2 - \frac{8}{\pi} \frac{\alpha^2}{\alpha^2 + \nu} \|\phi\|_{L^3}^3 \right). \end{aligned} \tag{5.6}$$

Consider some $\phi_1 \in H^1(\mathbb{R}^2)$ satisfying $\phi_1 \not\equiv 0$, and $0 \leq \phi_1 \leq \pi/4$ everywhere, and let $\phi_\lambda(x) = \phi_1(\lambda x)$. We have $\|\phi_\lambda\|_{L^2}^2 = \lambda^{-2} \|\phi_1\|_{L^2}^2$, $\|\phi_\lambda\|_{L^3}^3 = \lambda^{-2} \|\phi_1\|_{L^3}^3$

and $\|\nabla\phi_\lambda\|^2_{L^2} = \|\nabla\phi_1\|^2_{L^2}$. Then, for all $\alpha \in \mathbb{R}$, (5.6) implies

(5.7)
$$H(\alpha\phi_\lambda, \phi_\lambda) \leq \frac{\alpha^2+\nu}{4}\left(\|\nabla\phi_1\|^2_{L^2} + \frac{2q\lambda^{-2}}{\alpha^2+\nu}\|\phi_1\|^2_{L^2} - \frac{8}{\pi}\frac{\alpha^2\lambda^{-2}}{\alpha^2+\nu}\|\phi_1\|^3_{L^3}\right).$$

Fixing $\alpha > \left(\frac{\pi q\|\phi_1\|^2_{L2}}{4\|\phi_1\|^3_{L^3}}\right)^{1/2}$, we see from (5.7) that there exists $\lambda_0 > 0$ (depending on α, $\|\nabla\phi_1\|_{L^2}$) such that $0 < \lambda < \lambda_0$ implies $H(\alpha\phi_\lambda, \phi_\lambda) < 0$. on the other hand, $\|v\|^2 = \frac{\alpha^2}{\lambda^2}\|\phi_1\|^2_{L^2}$, therefore $a \geq b = \frac{\alpha^2}{\lambda_0^2}\|\phi_1\|^2_{L^2}$ implies $J_a < 0$. □

By Proposition 5.1 it is sufficient to look for the minimizer in $H^1_{\text{rad}}(\mathbb{R}^2) \times H^1_{\text{rad}}(\mathbb{R}^2)$.

Proposition 5.3. *Let $a > 0$ be such that $J_a < 0$. There exists $(v, \phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ that satisfies $(v, \phi) \in S_a$, and $H(v, \phi) = J_a$. In addition, we may assume that $(v, \phi) \in H^1_{\text{rad}}(\mathbb{R}^2) \times H^1_{\text{rad}}(\mathbb{R}^2)$, $v \geq 0$ and $\phi \in [0, \pi/4]$.*

Proof. Let $\mathcal{A} = \{(v_n, \phi_n)\}_{n\in\mathbb{Z}^+} \subset S_a$ be a minimizing sequence for H. By Lemma 5.1, and Proposition 5.1 we may assume that the minimizing sequence $\mathcal{A}$ also belongs to $H^1_{\text{rad}}(\mathbb{R}^2) \times H^1_{\text{rad}}(\mathbb{R}^2)$, and that its elements (v_n, ϕ_n) satisfy $v_n \geq 0$, $\phi_n \in [0, \pi/4]$ a.e. in $\mathbb{R}^2$, for all $n \in \mathbb{Z}^+$.

By (1.4) we have

$$\|v_n\|^2_{H^1} \leq \sup_n 4H(v_n, \phi_n) + 3a, \tag{5.8}$$

also, using $1 - cos(2\phi) \geq \phi^2$ for $\phi \in [0, \pi/4]$,

$$\begin{aligned} H(v, \phi) &\geq \frac{\nu}{4}\|\nabla\phi\|^2_{L^2} - \frac{1}{2}\|v\|^2_{L^2} + \frac{q}{4}\|\phi\|^2_{L^2} \\ &\geq -\frac{1}{2}a + \frac{1}{4}\min\{\nu, q\}\|\phi\|^2_{H^1}, \end{aligned}$$

therefore

$$\|\phi_n\|^2_{H^1} \leq C_{q,\nu}\left(\sup_n 4H(v_n, \phi_n) + 2a\right). \tag{5.9}$$

Then there exists a subsequence of $\mathcal{A}$ that is weakly convergent to (v, ϕ) in $H^1_{\text{rad}}(\mathbb{R}^2) \times H^1_{\text{rad}}(\mathbb{R}^2)$. We denote this subsequence also by $\mathcal{A} = \{(u_n, \phi_n)\}_{n\in\mathbb{Z}^+}$. Since $H^1_{\text{rad}}(\mathbb{R}^2)$ is compactly embedded in $L^p(\mathbb{R}^2)$, for any $2 < p < \infty$, see [3], the subsequence converges strongly to $(v, \phi) \in L^3(\mathbb{R}^2) \times L^3(\mathbb{R}^2)$. This implies that $v_n \to v$ and $\phi_n \to \phi$ a.e., therefore we may assume $v \geq 0$, $0 \leq \phi \leq \pi/4$ a.e. in $\mathbb{R}^2$.

To see that the limit is the minimizer, we use the weak semi-continuity of the L^2 norm to obtain

$$\|v\|_{L^2}^2 \le \liminf_n \|v_n\|_{L^2}^2 = a, \tag{5.10}$$

$$\frac{1}{4}\|\nabla v\|_{L^2}^2 + \frac{\nu}{4}\|\nabla \phi\|_{L^2}^2 \le \liminf_n \left(\frac{1}{4}\|\nabla v_n\|_{L^2}^2 + \frac{\nu}{4}\|\nabla \phi_n\|_{L^2}^2\right). \tag{5.11}$$

On the other hand, by Fatou's lemma,

$$\int_{\mathbb{R}^2} (1-\cos(2\phi))\,dx \le \liminf_n \int_{\mathbb{R}^2} (1-\cos(2\phi_n))\,dx. \tag{5.12}$$

Also, $v_n \to v$ in $L^3(\mathbb{R}^2)$ implies $v_n^2 \to v^2$ in $L^{3/2}(\mathbb{R}^2)$. Moreover, $|\sin(2\phi_n) - \sin(2\phi)| \le 2\,|\phi_n - \phi|$ implies $\sin(2\phi_n) \to \sin(2\phi)$ in $L^3(\mathbb{R}^2)$. Therefore

$$\lim_n \int_{\mathbb{R}^2} v_n^2 \sin\phi_n dx = \int_{\mathbb{R}^2} v^2 \sin\phi dx. \tag{5.13}$$

Collecting (5.11)-(5.13), we therefore have

$$H(v,\phi) \le \liminf_n H(u_n,\phi_n) = J_a < 0. \tag{5.14}$$

Using the fact that $H(v,0)$, $H(0,\phi) \ge 0$, we conclude that $(v,\phi) \not\equiv (0,0)$, $v \not\equiv 0$, moreover by (5.10) we have $0 < \|v\|_{L^2}^2 \le a$. Let $\lambda = \sqrt{a}/\|v\|_{L^2} \ge 1$, we check that $(\lambda\, v,\phi) \in S_a$, and that

$$J_a \le H(\lambda\, v,\phi) \le \lambda^2 H(v,\phi) \le \lambda^2 J_a \le J_a < 0. \tag{5.15}$$

It follows that $\lambda = 1$. Therefore $(v,\phi) \in S_a$, and $H(v,\phi) = J_a$. □

Corollary 5.1. *There exists $a_0 > 0$ such that $J_a = 0$ for $0 < a \le a_0$, and $J_a < 0$ for $a > a_0$. Moreover, the map $a \mapsto J_a$ is decreasing in (a_0,∞).*

Proof. Let $(v,\phi) \in S_a$ satisfying $H(v,\phi) = J_a < 0$, then for $\lambda > 1$ we have that $(\lambda\, v,\phi) \in S_{\lambda^2 a}$ and $H(\lambda\, v,\phi) \le \lambda^2 H(v,\phi) < 0$. Then if $0 < a < b$ we have

$$J_b \le \frac{b}{a} J_a < J_a < 0.$$

Defining $a_0 = \inf\{a > 0 : J_a < 0\}$, it therefore follows that $J_a < 0$ for all $a \in (a_0,\infty)$. By the definition of a_0 we have $J_a \ge 0$ for $a \in (0,a_0]$, and we use the scaling argument in the proof of Proposition 5.2 to conclude that $J_a = 0$. □

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A Derivation of model equations for optical solitons

We present the main steps of the derivation of model equations (1.1) from a "first-principles' model" that couples the Oseen-Frank equations for the nematic liquid crystal director field to Maxwell's equations for the electric fields, see [12, 7, 6]. The derivation is not rigorous, the goal is rather to make explicit the main assumptions leading to the simplified system of [16] and to (1.1) of [15].

The macroscopic average orientation of the liquid crystal molecules (director field) occupying a domain $\mathcal{D} \subset \mathbb{R}^3$ is described by the unit vector field $\mathbf{n} : \mathcal{D} \to S^2$. For suitably symmetric molecules we require that the image belongs to the projective plane, or equivalently that equations are symmetric under $\mathbf{n} \to -\mathbf{n}$.

The equations for the director field are obtained by formally extremizing the functional $\mathcal{V} + \mathcal{U}$, where $\mathcal{V}$ represents the Oseen-Frank elastic energy of the nematic liquid crystal, and $\mathcal{U}$ describes the coupling of the director field to external electric fields. The Oseen-Frank elastic energy $\mathcal{V}$ is given by

(A.1)
$$\mathcal{V} = \int_{\mathcal{D}} V, \quad V = \frac{1}{2}\left(K_1(\nabla \cdot \hat{\mathbf{n}})^2 + K_2(\hat{\mathbf{n}} \cdot \nabla \times \hat{\mathbf{n}})^2 + K_3|\hat{\mathbf{n}} \times \nabla \times \hat{\mathbf{n}}|^2\right),$$

with K_1, K_2, K_3 constants, see [7, 6]. V is the corresponding energy density. We will assume $K_1 = K_2 = K_3 = K$, see [15]. The coupling energy is

$$\mathcal{U} = -\int_{\mathcal{D}} U, \quad U = \mathbf{D} \cdot \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E}. \tag{A.2}$$

Physically, $\mathbf{D}$ is the displacement current, and ϵ the susceptibility matrix. We assume that ϵ at each point $\mathbf{x} \in \mathcal{D}$ is diagonal in a system of coordinates that consists of $\mathbf{n}(\mathbf{x})$, and two orthogonal vectors $\hat{e}_1(\mathbf{x})$, $\hat{e}_2(\mathbf{x})$ that are perpendicular to $\mathbf{n}(\mathbf{x})$. In the case where the molecules are symmetric with respect to rotations around $\mathbf{n}$ we have

(A.3)
$$\epsilon(\mathbf{x}) = \begin{bmatrix} \epsilon_{\parallel}(\mathbf{x}) & 0 & 0 \\ 0 & \epsilon_{\perp}(\mathbf{x}) & 0 \\ 0 & 0 & \epsilon_{\perp}(\mathbf{x}) \end{bmatrix}.$$

We then have

(A.4)
$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_{\parallel}(\mathbf{E}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \epsilon_{\perp}[\mathbf{E} - (\mathbf{E}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}] = (\epsilon_{\parallel} - \epsilon_{\perp})(\mathbf{E}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \epsilon_{\perp}\mathbf{E}.$$

For a homogeneous material, we can assume that the $\epsilon_{\parallel}$, $\epsilon_{\perp}$ are constant. By (A.2), (A.4) we then have

(A.5)
$$U = \delta\epsilon(\mathbf{E}\cdot\hat{\mathbf{n}})^2 + \epsilon_{\perp}(\mathbf{E}\cdot\mathbf{E})^2,$$

where $\delta\epsilon = \epsilon_{\parallel} - \epsilon_{\perp}$. We also assume $\delta\epsilon > 0$, this means that the molecule is oblong, and that $\mathbf{n}$ is the axis of the longer dimension.

The equations for $\mathbf{n}$ in the bulk are the formal Euler-Lagrange equations for $\mathcal{V}+\mathcal{U}$ in $\mathbb{R}^3$, assuming decay at infinity. Boundary conditions are imposed afterwards. For instance, describing $\mathbf{n}(\mathbf{x})$ by polar and azimuthal angles $\phi_1(\mathbf{x})$, $\phi_2(\mathbf{x})$, $\mathbf{x} = [x_1, x_2, x_3]$, the Euler-Langrange equations are

(A.6)
$$\frac{\partial L}{\partial \phi_k} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\frac{\partial L}{\partial \phi_{k,\,j}} = 0,$$
$$\text{with} \quad L = V - U, \quad \phi_{k,\,j} = \frac{\partial \phi_k}{\partial x_j}, \quad k = 1, 2.$$

In the experimental geometry of interest, see [4], the nematic liquid crystal occupies the region between two parallel planes at $x = \pm d/2$, i.e x is the vertical coordinate. The horizontal coordinate z is referred to as the *optical axis*. The coordinate y represents the second direction that is transverse to the optical axis. Thus $\mathcal{D} = \{(x,y,z) \in \mathbb{R}^3 : (y,z) \in \mathbb{R}^2, x \in [-d/2, d/2]\}$.

We shall further assume that the director and electric fields have the respective form

(A.7)
$$\mathbf{n}(x) = [\sin\vartheta(z,y,x), 0, \cos\vartheta(x,y,z)], \quad \mathbf{E} = [E(x,y,z), 0, 0].$$

Physically, light is polarized on the plane of the director field vectors, and is perpendicular to the optical axis, see [15] for more general polarizations.

By (A.7), (A.5) the energies of (A.1), (A.2) reduce to

(A.8)
$$V = \frac{1}{2}K(\vartheta_x^2 + \vartheta_y^2 + \vartheta_z^2), \quad U = \delta\epsilon(E)^2\sin^2 + \epsilon_{\perp}E^2.$$

Then by (A.6),

$$K(\vartheta_{xx} + \vartheta_{yy} + \vartheta_{zz}) = -\delta\epsilon(E)^2 \sin 2\vartheta. \tag{A.9}$$

Note that in the above formulation we have assumed that all quantities are independent of time. We will however use equation (A.9) also for cases where the electric field varies rapidly, with $(E)^2$ replaced in (A.9) by a suitable time average. In particular, we are interested in E of the form $E = E_b + E_L$, where E_b is a constant field applied to the sample, and $E_L(x, y, z, t)$ represents the field of a laser beam that we shine from the left side. Letting $E_L(x, y, z, t) = \mathrm{Re}\,(A(x, y, z)e^{i\omega t})$, with $A(x, y, z)$ a complex amplitude specified below, and replace (A.9) by

$$K(\nabla^2\vartheta + \vartheta_{zz}) = -\delta\epsilon(E_b^2 + \frac{1}{2}|A|^2) \sin 2\vartheta, \tag{A.10}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2$ is the transverse Laplacian. Equation (A.10) is in essence the second equation of (1.2) in [12], see also [7]. The difference is that we here have an additional constant field E_b.

The constant E_b is assumed to satisfy $E_b \geq \mathcal{E}_0$, where $\mathcal{E}_0$ is the smallest $\mathcal{E} > 0$ for which

$$K\Theta_{xx} = -\delta\epsilon\mathcal{E}^2 \sin 2\Theta, \quad \Theta(-d/2) = \Theta(d/2) = 0, \tag{A.11}$$

has a nontrivial solution $\Theta : [-d/2, d/2] \to \mathbb{R}$.

$\mathcal{E}_0$ is known as the Friedriekcz threshold field (for $[-d/2, d/2]$). Given $\mathcal{E} \geq \mathcal{E}_0$, (A.11) has exactly two nontrivial solutions $\pm\Theta(\mathcal{E})$, with $\Theta(\mathcal{E})(x) > 0$, $\forall x \in (-d/2, d/2)$. The maximum of $\Theta(\mathcal{E})$ is attained at $x = 0$, see e.g. [15].

Physically, the boundary condition in (A.11) means that the director field is fixed at the vertical physical boundaries of the sample, and that it is parallel to the walls, in the required plane. Moreover, the bias electric field E_b should be strong enough to start rotating the molecules inside the sample. If $\mathcal{E} < \mathcal{E}_0$ then the only solution of (A.11) is the trivial one. In the experiments modeled by [12], the boundary conditions are the same but the constant term E_b is absent. The laser field $|A|^2$ must be strong enough to overcome this barrier to rotate the molecules. This leads to a more complicated bifurcation problem.

We now turn to Maxwell's equations for the electromagnetic field inside the liquid crystal sample. As before, we derive the bulk equations first. Maxwell's equations for a dielectric in $\mathbb{R}^3$ are

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0, \quad \nabla \times B - \frac{1}{c}\frac{\partial D}{\partial t} = 0, \quad \nabla \cdot \mathbf{B} = 0, \tag{A.12}$$

where $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$ are vector fields in $\mathbb{R}^3$, representing the displacement current, and the electric and magnetic fields respectively. We immediately obtain

$$c^2(\nabla^2\mathbf{E} + \mathbf{E}_{zz}) - \frac{\partial^2\mathbf{D}}{\partial t^2} = 0. \tag{A.13}$$

The relation $\mathbf{D} = \epsilon\mathbf{E}$, and (A.4) couples the electric field $\mathbf{E}$ to $\mathbf{n}$. Note that (A.13) describes three equations, corresponding to the components of $\mathbf{E}$, and $\mathbf{D}$ along the x, y and z axes. We further assume (A.7), which reduces (A.13) to two equations for the x and $z-$components respectively. Following [12], we will keep only the equation for the $x-$axis components of $\mathbf{E}$, $\mathbf{D}$ in (A.13), namely

$$c^2(\nabla^2 E + E_{zz}) - \frac{\partial^2}{\partial t^2}(\delta\epsilon E\sin^2\vartheta + \epsilon_\perp E) = 0. \tag{A.14}$$

A further simplification comes by assuming that since $\mathbf{n}$ in (A.10) is an equation for an averaged director field, then we can write $\partial_t^2(E\sin^2\vartheta) \approx \sin^2\vartheta\partial_t^2 E$, so that A.14 is further simplified to

$$\nabla^2 E + E_{zz} - \frac{n_\perp^2}{c^2}(\alpha\sin^2\vartheta + 1)\partial_t^2 E = 0, \tag{A.15}$$

where have introduced the parallel and perpendicular refraction indices $n_\parallel^2 = \epsilon_\parallel$, $n_\perp^2 = \epsilon_\perp$ respectively, and the dielectric anisotropy parameter $\alpha = (\epsilon_\parallel - \epsilon_\perp)/\epsilon_\perp$. Equation (A.15) is analogous to the first equation of (1.2) in [12]. We consider an electric field of the form

$$E(x, y, z, t) = E_b + A(\sqrt{\varepsilon}x, \sqrt{\varepsilon}y, \varepsilon z, \varepsilon t)e^{\mathrm{i}(kz-\omega t)} + \text{c.c.}, \tag{A.16}$$

where k, ω are related through

$$k^2 = \frac{n_\perp^2}{c^2}(\alpha\sin^2\theta_0 + 1)\omega^2, \tag{A.17}$$

and θ_0 is the maximum value of a solution Θ_0 of (A.11) with $\mathcal{E} = E_b > \mathcal{E}_0$. (The maximum is attained at the origin.) The bias field E_b is constant and the second term in (A.16) describes the laser field. Letting $x_1 = \sqrt{\varepsilon}x$, $y_1 = \sqrt{\varepsilon}y$, $z_1 = \varepsilon z$, $t_1 = \varepsilon t$, and $\nabla_1^2 = \partial_{x_1}^2 + \partial_{y_1}^2$, (A.15), with A.16 leads to

$$\varepsilon\nabla_1^2 A + 2\mathrm{i}k\varepsilon\partial_{z_1}A + \frac{n_\perp^2}{c^2}\alpha(\sin^2\vartheta - \sin^2\theta_0)A + 2i\omega\frac{n_\perp^2}{c^2}\varepsilon\partial_{t_1}A = O(2), \tag{A.18}$$

where $O(2)$ denotes terms that are of order 2 or higher in ε and α. By (A.17) we have $\omega = \pm\frac{c}{n_\perp}k + O(\alpha)$, so that letting $\tau = t_1 \mp \frac{n_\perp}{c}z_1$, $\zeta = z_1$, (A.18) becomes the NLS-type equation

$$2ik\varepsilon\partial_\zeta A + \varepsilon\nabla_1^2 A + \frac{n_\perp^2}{c^2}\alpha(\sin^2\vartheta - \sin^2\theta_0)A = 0, \tag{A.19}$$

up to an error of $O(2)$. Using the scaled variables x_1, y_1, z_1 in (A.10) we also obtain

$$K\varepsilon\nabla_1^2\vartheta = -\alpha n_\perp^2(E_b^2 + \frac{1}{2}|A|^2)\sin 2\vartheta, \tag{A.20}$$

up to an error of $O(2)$. This step eliminates the derivatives along z_1, up to $O(2)$.

Equations (A.19), (A.20) originally appeared in [16]. We note that there is no assumption on the size of the laser field relative to E_b. The derivation for $E_b = 0$ is the same, but θ_0 must be interpreted as a "typical value" of ϑ in the region of interest.

We derive from (A.19), (A.20) a second system by letting $\vartheta = \Theta_0 + \theta$, where ϑ, and θ are functions of x_1, y_1, z_1, and $\Theta_0(x)$ satisfies (A.11) with $\mathcal{E} = E_b$, $E_b \geq \mathcal{E}$, Thus θ is the deviation from the "pre-tilt" angle Θ_0.

A first step is to consider the director equation (A.20) and use equation (A.11) for Θ_0, obtaining

$$\begin{aligned} K\nabla^2\theta = {} & \alpha n_\perp^2(E_b^2\sin 2\Theta_0(1-\cos 2\theta) - \frac{1}{2}|A|^2\cos 2\Theta_0\sin 2\theta \\ & - E_b^2\cos 2\Theta_0\sin 2\theta - \frac{1}{2}|A|^2\sin 2\Theta_0\cos 2\theta). \end{aligned} \tag{A.21}$$

We simplify this equation by assuming $\Theta_0 \approx \theta_0$. Letting $q = \theta_0 - \pi/4$ we further assume that $q \sim |A|/E_b \sim h$, $\theta \sim h^2$, where h is a small parameter. (A.21) then becomes

$$K\nabla^2\theta = -\alpha n_\perp^2 E_b^2\cos 2\theta_0\sin 2\theta - \alpha n_\perp^2\frac{1}{2}|A|^2\sin 2\theta_0\cos 2\theta, \tag{A.22}$$

up to an error of $O(h^4)$. The two terms we kept were $\sim h^3$ and $\sim h^2$. We also assume $q > 0$, so that $\cos 2\theta_0 = -2q + O(q^3) < 0$ in (A.22). Using similarly $\Theta_0 \approx \theta_0$, and the same scaling assumptions on q, θ in (A.19) we have

$$2i\varepsilon k\partial_\zeta A + \nabla^2 A + \frac{n_\perp^2}{c^2}\alpha\sin 2\theta_0\sin 2\theta A = 0, \tag{A.23}$$

up to an error of $O(h^4)$. The term we kept is $\sim h^2$. System (A.22), (A.23) leads to (1.1).

The derivation of system (A.22), (A.23) therefore assumes that the laser field is small compared to the bias field. Also the additional angle θ is assumed small, while $q > 0$ and small implies that the laser beam goes through a region of the sample where the pre-tilt angle θ_0 is slightly above $\pi/4$. The above derivation also suggests more general systems, e.g. using

(A.21) for the director field, with Θ_0 suitably approximated in an infinite domain, e.g. $\Theta_0 = \theta_0$. The infinite domain problem approximates small scale effects in a region near $x = 0$. Equations (A.19), (A.20) require a more careful analysis of the to scales of the fields produced by E_b and $|A|$.

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